Les limites

Exercice 1

a) $\lim\limits_{\substack{x\to 0\\<}} f(x) = 1$ $\lim\limits_{\substack{x\to 0\\>}} f(x) = 1$ $\lim\limits_{x\to 0} f(x) = 1$

b) $\lim\limits_{\substack{x\to 0\\<}} f(x) = -\infty$ $\lim\limits_{\substack{x\to 0\\>}} f(x) = +\infty$ $\lim\limits_{x\to 0} f(x)$ n'existe pas

c) $\lim\limits_{\substack{x\to 0\\<}} f(x) = 1$ $\lim\limits_{\substack{x\to 0\\>}} f(x) = 0$ $\lim\limits_{x\to 0} f(x)$ n'existe pas

d) $\lim\limits_{\substack{x\to 0\\<}} f(x) = 0$ $\lim\limits_{\substack{x\to 0\\>}} f(x) = 0$ $\lim\limits_{x\to 0} f(x) = 0$

Exercice 2

a) $\lim\limits_{x\to 0} \dfrac{x^2-2x}{x} = \dfrac{0}{0} \quad \Rightarrow \quad \lim\limits_{x\to 0} \dfrac{x^2-2x}{x} = \lim\limits_{x\to 0} \dfrac{x(x-2)}{x} = \lim\limits_{x\to 0} x-2 = -2$

b) $\lim\limits_{x\to -1} \dfrac{2x^2+x-1}{x^3+1} = \dfrac{0}{0}$

$$\lim_{x\to -1} \frac{2x^2+x-1}{x^3+1} = \lim_{x\to -1} \frac{(2x-1)(x+1)}{(x+1)(x^2-x+1)} = \lim_{x\to -1} \frac{2x-1}{x^2-x+1} = \frac{-3}{3} = -1$$

c) $\lim\limits_{x\to 0} \dfrac{3x^2-2x+2}{x+1} = \dfrac{2}{1} = 2$

d) $\lim\limits_{x\to 3} \dfrac{x^2+2x-15}{x^2+8x+15} = \dfrac{0}{48} = 0$

e) $\lim\limits_{x\to -3} \dfrac{x^2+2x-15}{x^2+8x+15} = \dfrac{-12}{0}$

$$\lim_{\substack{x\to -3\\<}} \frac{x^2+2x-15}{x^2+8x+15} = \lim_{\substack{x\to -3\\<}} \frac{-12}{\to 0^-} = +\infty$$

$$\lim_{\substack{x\to -3\\>}} \frac{x^2+2x-15}{x^2+8x+15} = \lim_{\substack{x\to -3\\>}} \frac{-12}{\to 0^+} = -\infty$$

f) $\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x^2-4x+4} = \dfrac{0}{0}$

$\lim\limits_{x\to 2} \dfrac{x^2-3x+2}{x^2-4x+4} = \lim\limits_{x\to 2} \dfrac{(x-2)(x-1)}{(x-2)^2} = \lim\limits_{x\to 2} \dfrac{x-1}{x-2} = \dfrac{1}{0}$

$\lim\limits_{\substack{x\to 2\\<}} \dfrac{x^2-3x+2}{x^2-4x+4} = \lim\limits_{\substack{x\to 2\\<}} \dfrac{x-1}{x-2} = \dfrac{1}{\to 0^-} = -\infty$

$\lim\limits_{\substack{x\to 2\\>}} \dfrac{x^2-3x+2}{x^2-4x+4} = \lim\limits_{\substack{x\to 2\\>}} \dfrac{x-1}{x-2} = \dfrac{1}{\to 0^+} = +\infty$

g) $\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} = \dfrac{0}{0}$

$\lim\limits_{x\to 4} \dfrac{\sqrt{x}-2}{x-4} = \lim\limits_{x\to 4} \dfrac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \lim\limits_{x\to 4} \dfrac{x-4}{(x-4)(\sqrt{x}+2)} = \lim\limits_{x\to 4} \dfrac{1}{\sqrt{x}+2} = \dfrac{1}{4}$

h) $\lim\limits_{x\to 5} \dfrac{x-5}{\sqrt{2x-1}-3} = \dfrac{0}{0}$

$\lim\limits_{x\to 5} \dfrac{x-5}{\sqrt{2x-1}-3} = \lim\limits_{x\to 5} \dfrac{(x-5)(\sqrt{2x-1}+3)}{(\sqrt{2x-1}-3)(\sqrt{2x-1}+3)} = \lim\limits_{x\to 5} \dfrac{(x-5)(\sqrt{2x-1}+3)}{2x-10} =$
$\lim\limits_{x\to 5} \dfrac{\sqrt{2x-1}+3}{2} = \dfrac{6}{2} = 3$

i) $\lim\limits_{x\to 11} \dfrac{3-\sqrt{x-2}}{x-11} = \dfrac{0}{0}$

$\lim\limits_{x\to 11} \dfrac{3-\sqrt{x-2}}{x-11} = \lim\limits_{x\to 11} \dfrac{(3-\sqrt{x-2})(3+\sqrt{x-2})}{(x-11)(3+\sqrt{x-2})} = \lim\limits_{x\to 11} \dfrac{(11-x}{(x-11)(3+\sqrt{x-2})} =$
$\lim\limits_{x\to 11} \dfrac{-1}{(3+\sqrt{x-2})} = -\dfrac{1}{6}$

j) $\lim\limits_{x\to +\infty} \dfrac{6x^4-3x^2+2}{x^3-27} = \lim\limits_{x\to +\infty} \dfrac{6x^4}{x^3} = \lim\limits_{x\to +\infty} 6x = +\infty$

k) $\lim\limits_{x\to -\infty} \dfrac{3x-2}{9x+7} = \lim\limits_{x\to -\infty} \dfrac{3x}{9x} = \lim\limits_{x\to -\infty} \dfrac{1}{3} = \dfrac{1}{3}$

l) $\lim\limits_{x\to +\infty} \dfrac{x^2+x-2}{4x^3-1} = \lim\limits_{x\to +\infty} \dfrac{x^2}{4x^3} = \lim\limits_{x\to +\infty} \dfrac{1}{4x} = 0$

Exercice 3

a) $a(x) = \dfrac{x^2+x-12}{x^2-3x-10} = \dfrac{(x+4)(x-3)}{(x-5)(x+2)}$ $\qquad ED(a) = \mathbb{R} - \{-2;5\}$

AV:

$$\lim_{x\to -2} \frac{x^2+x-12}{x^2-3x-10} = \frac{-10}{0}$$

$$\lim_{\substack{x\to -2\\<}} \frac{x^2+x-12}{x^2-3x-10} = \lim_{\substack{x\to -2\\<}} \frac{-10}{\to 0^+} = -\infty \qquad \lim_{\substack{x\to -2\\>}} \frac{x^2+x-12}{x^2-3x-10} = \lim_{\substack{x\to -2\\>}} \frac{-10}{\to 0^-} = +\infty$$

$$\lim_{x\to 5} \frac{x^2+x-12}{x^2-3x-10} = \frac{18}{0}$$

$$\lim_{\substack{x\to 5\\<}} \frac{x^2+x-12}{x^2-3x-10} = \lim_{\substack{x\to 5\\<}} \frac{18}{\to 0^-} = -\infty \qquad \lim_{\substack{x\to 5\\>}} \frac{x^2+x-12}{x^2-3x-10} = \lim_{\substack{x\to 5\\>}} \frac{18}{\to 0^+} = +\infty$$

$\boxed{\text{asymptotes verticales: } x=-2 \text{ et } x=5}$

AH:

$$\lim_{x\to -\infty} \frac{x^2+x-12}{x^2-3x-10} = \lim_{x\to -\infty} \frac{x^2}{x^2} = \lim_{x\to -\infty} 1 = 1$$

$$\lim_{x\to +\infty} \frac{x^2+x-12}{x^2-3x-10} = \lim_{x\to +\infty} \frac{x^2}{x^2} = \lim_{x\to +\infty} 1 = 1$$

$\boxed{\text{asymptote horizontale: } y=1}$

position de l'asymptote $y=1$ par rapport à la courbe $y=a(x)$

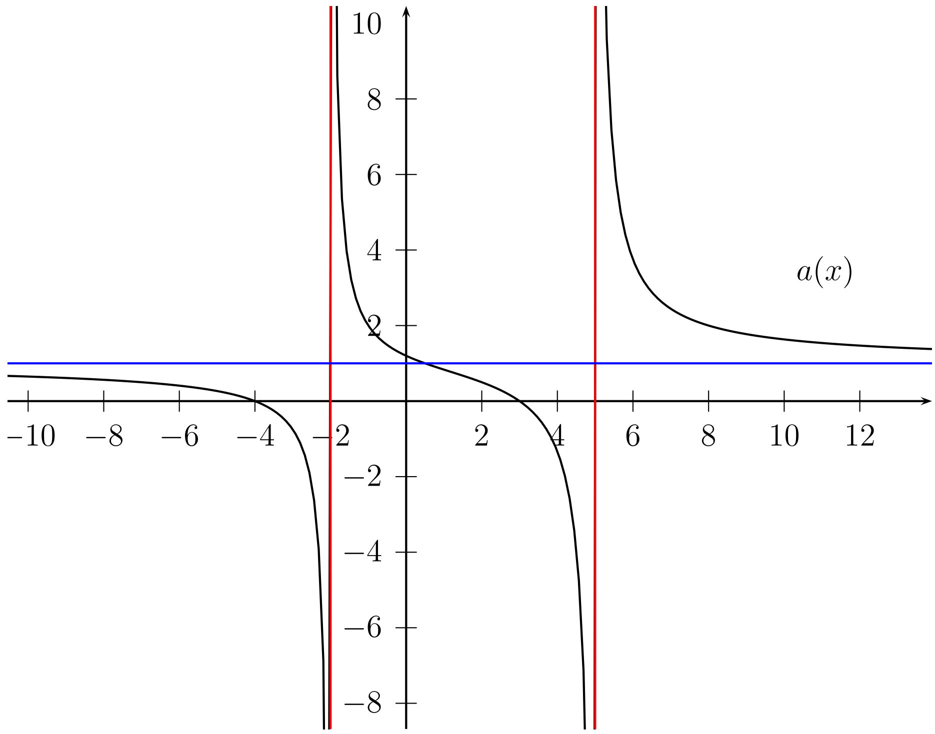
$$a(x)-1 = \frac{x^2+x-12}{x^2-3x-10} - 1 = \frac{4x-2}{x^2-3x-10}$$

x	$-\infty$		-2		$\frac{1}{2}$		5		$+\infty$
$4x-2$		$-$	|	$-$	0	$+$	|	$+$	
$x^2-3x-10$		$+$	0	$-$	|	$-$	0	$+$	
$a(x)-1$		$-$	‖	$+$	0	$-$	‖	$+$	

la courbe $y=a(x)$ est au-dessus de l'AH si $x \in]-2;\frac{1}{2}[\cup]5;+\infty[$

la courbe $y=a(x)$ est au-dessous de l'AH si $x \in]-\infty;-2[\cup]\frac{1}{2};5[$

la courbe $y=a(x)$ coupe l'AH au point $(\frac{1}{2};1)$



b) $b(x) = \dfrac{2(x+2)^2}{(x-3)(x+1)}$ $\qquad ED(b) = \mathbb{R} - \{-1; 3\}$

AV:

$$\lim_{x \to -1} \frac{2(x+2)^2}{(x-3)(x+1)} = \frac{2}{0}$$

$$\lim_{\substack{x \to -1 \\ <}} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{\substack{x \to -1 \\ <}} \frac{2}{\to 0^+} = +\infty \qquad \lim_{\substack{x \to -1 \\ >}} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{\substack{x \to -1 \\ >}} \frac{2}{\to 0^-} = -\infty$$

$$\lim_{x \to 3} \frac{2(x+2)^2}{(x-3)(x+1)} = \frac{50}{0}$$

$$\lim_{\substack{x \to 3 \\ <}} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{\substack{x \to 3 \\ <}} \frac{50}{\to 0^-} = -\infty \qquad \lim_{\substack{x \to 3 \\ >}} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{\substack{x \to 3 \\ >}} \frac{50}{\to 0^+} = +\infty$$

$\boxed{\text{asymptotes verticales: } x = -1 \text{ et } x = 3}$

AH:

$$\lim_{x \to -\infty} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{x \to -\infty} \frac{2x^2}{x^2} = \lim_{x \to -\infty} 2 = 2$$

$$\lim_{x \to +\infty} \frac{2(x+2)^2}{(x-3)(x+1)} = \lim_{x \to +\infty} \frac{2x^2}{x^2} = \lim_{x \to +\infty} 2 = 2$$

$\boxed{\text{asymptote horizontale: } y = 2}$

position de l'asymptote $y = 2$ par rapport à la courbe $y = b(x)$

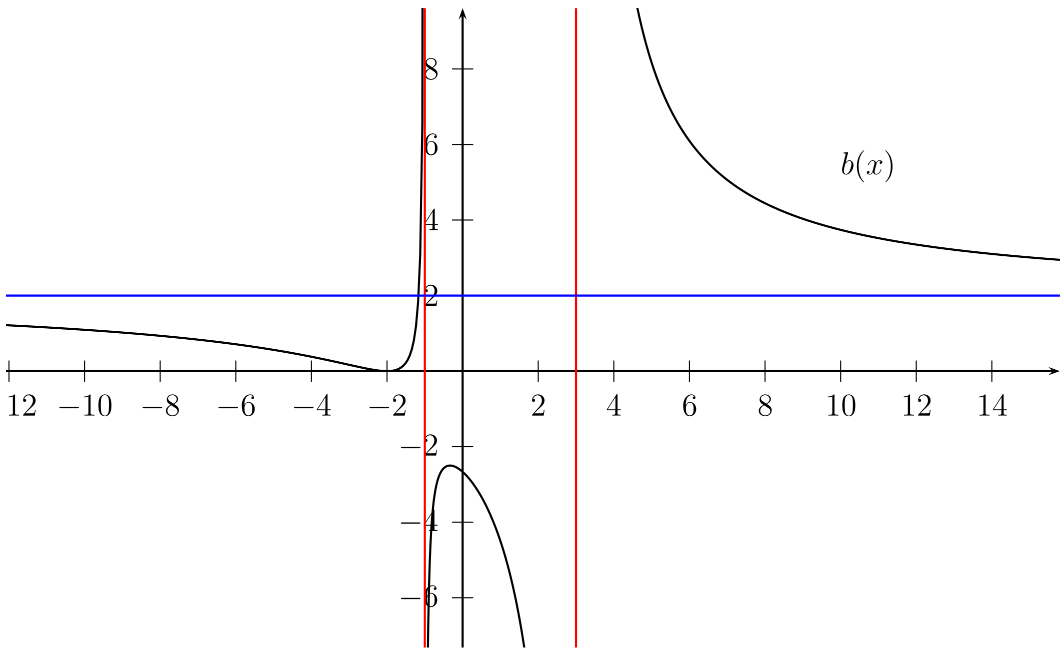
$$b(x) - 2 = \frac{2(x+2)^2}{x^2 - 2x - 3} - 2 = \frac{12x + 14}{x^2 - 2x - 3}$$

x	$-\infty$		$-\frac{7}{6}$		-1		3		$+\infty$
$12x+14$		$-$	0	$+$		$+$		$+$	
x^2-2x-3		$+$		$+$	0	$-$	0	$+$	
$b(x)-2$		$-$	0	$+$	$\|$	$-$	$\|$	$+$	

la courbe $y = b(x)$ est au-dessus de l'AH si $x \in]-\frac{7}{6}; -1[\cup]3; +\infty[$

la courbe $y = b(x)$ est au-dessous de l'AH si $x \in]-\infty; -\frac{7}{6}[\cup]-1; 3[$

la courbe $y = b(x)$ coupe l'AH au point $(-\frac{7}{6}; 2)$



c) $c(x) = \dfrac{x^2-1}{x^3}$ $\qquad ED(c) = \mathbb{R}^*$

AV:

$$\lim_{x\to 0} \frac{x^2-1}{x^3} = \frac{-1}{0}$$

$$\lim_{\substack{x\to 0\\ <}} \frac{x^2-1}{x^3} = \lim_{\substack{x\to 0\\ <}} \frac{-1}{\to 0^-} = +\infty \qquad \lim_{\substack{x\to 0\\ >}} \frac{x^2-1}{x^3} = \lim_{\substack{x\to 0\\ >}} \frac{-1}{\to 0^+} = -\infty$$

$\boxed{\text{asymptote verticale: } x = 0}$

AH:

$$\lim_{x\to -\infty} \frac{x^2-1}{x^3} = \lim_{x\to -\infty} \frac{x^2}{x^3} = \lim_{x\to -\infty} \frac{1}{x} = 0$$

$$\lim_{x\to +\infty} \frac{x^2-1}{x^3} = \lim_{x\to +\infty} \frac{x^2}{x^3} = \lim_{x\to +\infty} \frac{1}{x} = 0$$

asymptote horizontale: $y = 0$

position de l'asymptote $y = 0$ par rapport à la courbe $y = c(x)$

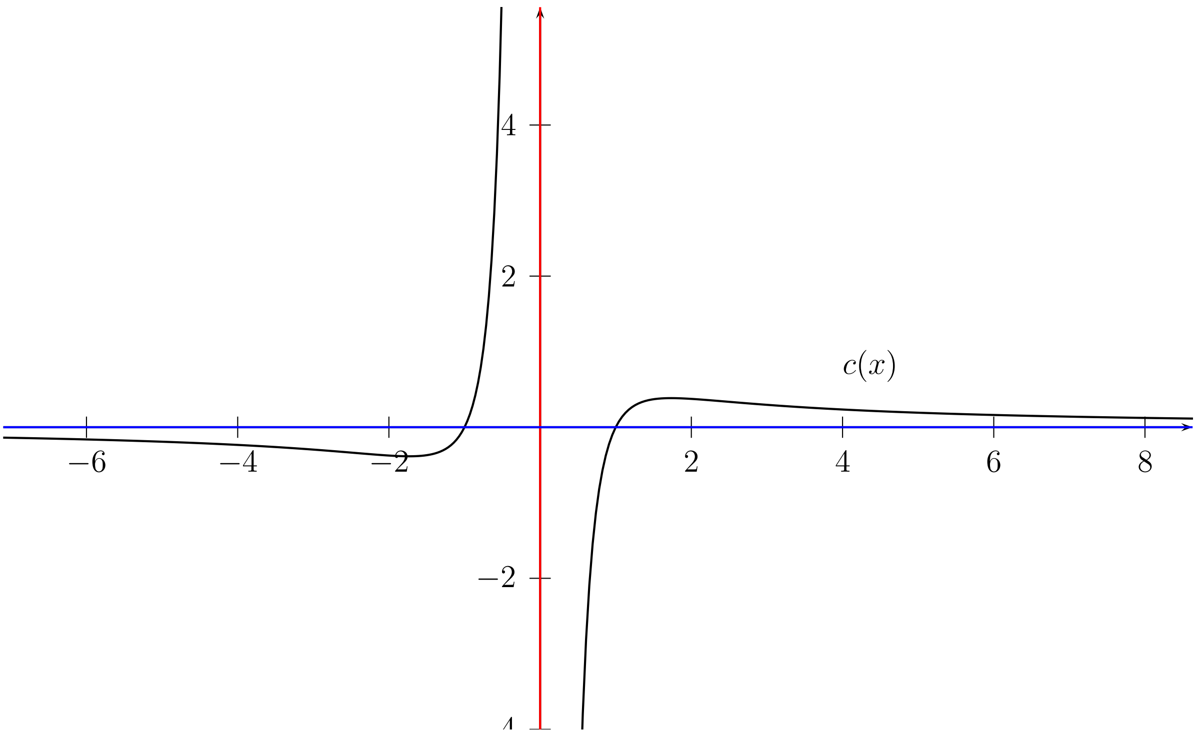
$c(x) - 0 = \dfrac{x^2 - 1}{x^3}$

x	$-\infty$		-1		0		1		$+\infty$
$x^2 - 1$		$+$	0	$-$	|	$-$	0	$+$	
x^3		$-$	|	$-$	0	$+$	|	$+$	
$b(x) - 2$		$-$	0	$+$	||	$-$	|	$+$	

la courbe $y = c(x)$ est au-dessus de l'AH si $x \in]-1; 0[\cup]1; +\infty[$

la courbe $y = c(x)$ est au-dessous de l'AH si $x \in]-\infty; -1[\cup]0; 1[$

la courbe $y = c(x)$ coupe l'AH aux points $(-1; 0)$ et $(1; 0)$



d) $d(x) = \dfrac{x^2-1}{x}$ $\qquad ED(d) = \mathbb{R}^*$

AV:

$\lim\limits_{x\to 0} \dfrac{x^2-1}{x} = \dfrac{-1}{0}$

$\lim\limits_{\substack{x\to 0\\<}} \dfrac{x^2-1}{x} = \lim\limits_{\substack{x\to 0\\<}} \dfrac{-1}{\to 0^-} = +\infty \qquad \lim\limits_{\substack{x\to 0\\>}} \dfrac{x^2-1}{x} = \lim\limits_{\substack{x\to 0\\>}} \dfrac{-1}{\to 0^+} = -\infty$

asymptote verticale: $x = 0$

AO:

$\dfrac{x^2-1}{x}$ donne un quotient $Q = x$ avec un reste $R = -1$

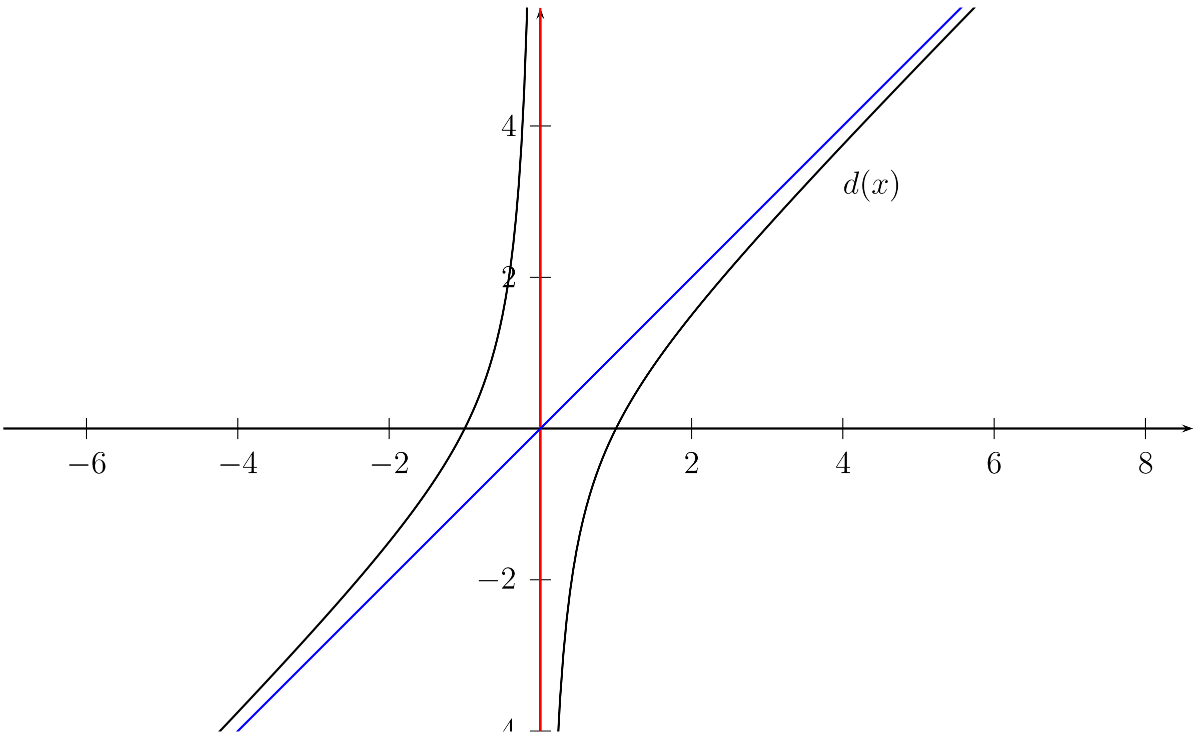
asymptote oblique: $y = x$

$d(x) - x = \dfrac{x^2-1}{x} - x = \dfrac{-1}{x}$

x	$-\infty$	0	$+\infty$
-1	$-$		$-$
x	$-$	0	$+$
$d(x)-x$	$+$	‖	$-$

la courbe $y = d(x)$ est au-dessus de l'AO si $x \in]-\infty; 0[$

la courbe $y = d(x)$ est au-dessous de l'AO si $x \in]0; +\infty[$



e) $e(x) = \dfrac{x^3+12x^2-15x}{x^2+x} = \dfrac{x^3+12x^2-15x}{x(x+1)}$ $\qquad ED(e) = \mathbb{R}^* - \{-1\}$

AV:

$$\lim_{x\to -1} \frac{x^3+12x^2-15x}{x^2+x} = \frac{26}{0}$$

$$\lim_{\substack{x\to -1 \\ <}} \frac{x^3+12x^2-15x}{x^2+x} = \lim_{\substack{x\to -1 \\ <}} \frac{26}{\to 0^+} = +\infty$$

$$\lim_{\substack{x\to -1 \\ >}} \frac{x^3+12x^2-15x}{x^2+x} = \lim_{\substack{x\to -1 \\ >}} \frac{26}{\to 0^-} = -\infty$$

$$\lim_{x\to 0} \frac{x^3+12x^2-15x}{x^2+x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{x^3+12x^2-15x}{x^2+x} = \lim_{x\to 0} \frac{x(x^2+12x-15)}{x(x+1)} = \lim_{x\to 0} \frac{x^2+12x-15}{x+1} = -15 \text{ (trou)}$$

$\boxed{\text{asymptote verticale: } x = -1}$

AO:

$\dfrac{x^3+12x^2-15x}{x^2+x}$ donne un quotient $Q = x + 11$ avec un reste $R = -26x$

$\boxed{\text{asymptote oblique: } y = x + 11}$

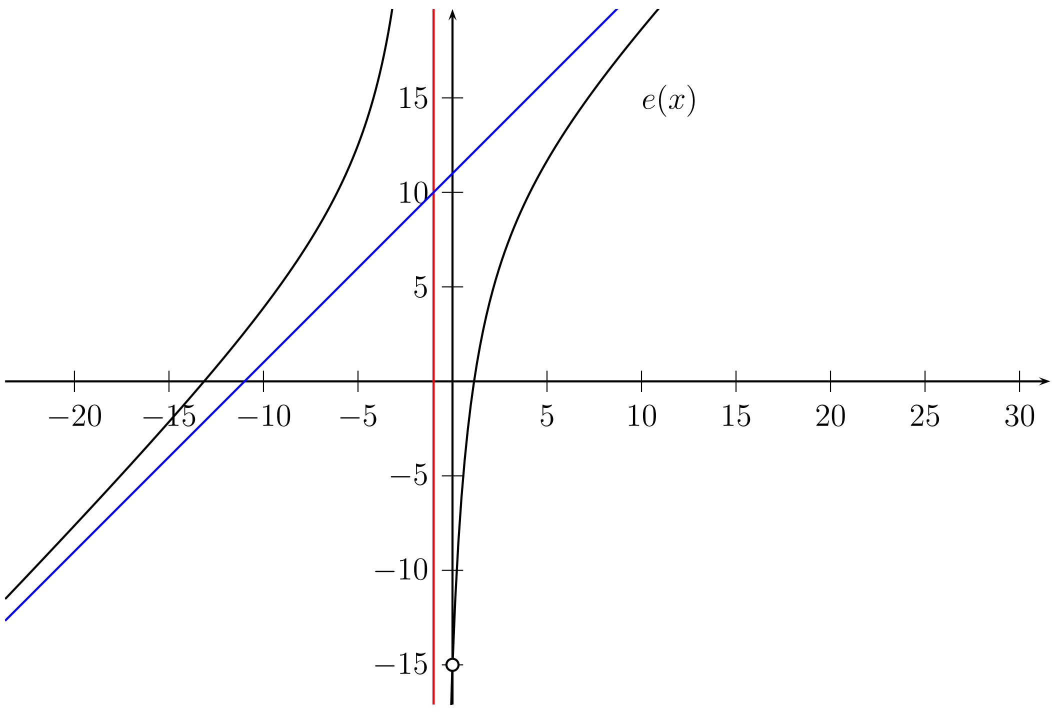
position de l'asymptote $y = x + 11$ par rapport à la courbe $y = e(x)$

$$e(x) - (x+11) = \frac{x^3+12x^2-15x}{x^2+x} - (x+11) = \frac{-26x}{x^2+x}$$

x	$-\infty$		-1		0		$+\infty$
$-26x$		$+$		$+$	0	$-$	
x^2+x		$+$	0	$-$	0	$+$	
$e(x)-(x+11)$		$+$	‖	$-$	‖	$-$	

la courbe $y = e(x)$ est au-dessus de l'AO si $x \in]-\infty; -1[$

la courbe $y = e(x)$ est au-dessous de l'AO si $x \in]-1; \infty[$



f) $f(x) = \dfrac{x^3+x^2+2x}{x^2+3x} = \dfrac{x^3+x^2+2x}{x(x+3)}$ $\qquad ED(f) = \mathbb{R}^* - \{-3\}$

AV:

$$\lim_{x\to -3} \frac{x^3+x^2+2x}{x^2+3x} = \frac{-24}{0}$$

$$\lim_{\substack{x\to -3\\ <}} \frac{x^3+x^2+2x}{x^2+3x} = \lim_{\substack{x\to -3\\ <}} \frac{-24}{\to 0^+} = -\infty$$

$$\lim_{\substack{x\to -3\\ >}} \frac{x^3+x^2+2x}{x^2+3x} = \lim_{\substack{x\to -3\\ >}} \frac{-24}{\to 0^-} = +\infty$$

$$\lim_{x\to 0} \frac{x^3+x^2+2x}{x^2+3x} = \frac{0}{0}$$

$$\lim_{x\to 0} \frac{x^3+x^2+2x}{x^2+3x} = \lim_{x\to 0} \frac{x(x^2+x+2)}{x(x+3)} = \lim_{x\to 0} \frac{x^2+x+2}{x+3} = \frac{2}{3} \text{ (trou)}$$

$\boxed{\text{asymptote verticale: } x = -3}$

AO:

$\dfrac{x^3+x^2+2x}{x^2+3x}$ donne un quotient $Q = x-2$ avec un reste $R = 8x$

$\boxed{\text{asymptote oblique: } y = x-2}$

position de l'asymptote $y = x-2$ par rapport à la courbe $y = f(x)$

$$f(x) - (x-2) = \frac{x^3+x^2+2x}{x^2+3x} - (x-2) = \frac{8x}{x^2+3x}$$

x	$-\infty$		-3		0		$+\infty$
$8x$		$-$		$-$	0	$+$	
x^2+3x		$+$	0	$-$	0	$+$	
$f(x)-(x-2)$		$-$	‖	$+$	‖	$+$	

la courbe $y = f(x)$ est au-dessus de l'AO si $x \in]-3; +\infty[$

la courbe $y = f(x)$ est au-dessous de l'AO si $x \in]-\infty; -3[$