Dérivée

Exercice 1

a) $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{x^2-x-a^2+a}{x-a} = \dfrac{0}{0}$

$\lim\limits_{x\to a} \dfrac{(x^2-a^2)-(x-a)}{x-a} = \lim\limits_{x\to a} \dfrac{(x+a)(x-a)-1(x-a)}{x-a} = \lim\limits_{x\to a} \dfrac{(x-a)(x+a-1)}{x-a} =$
$\lim\limits_{x\to a} x+a-1 = 2a-1$

$\Rightarrow \boxed{f'(x) = 2x-1}$

b) $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{x^2+3x+2-a^2-3a-2}{x-a} = \dfrac{0}{0}$

$\lim\limits_{x\to a} \dfrac{(x^2-a^2)+(3x-3a)}{x-a} = \lim\limits_{x\to a} \dfrac{(x+a)(x-a)+3(x-a)}{x-a} = \lim\limits_{x\to a} \dfrac{(x-a)(x+a+3)}{x-a} =$
$\lim\limits_{x\to a} x+a+3 = 2a+3$

$\Rightarrow \boxed{f'(x) = 2x+3}$

c) $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{\frac{1}{x}-\frac{1}{a}}{x-a} = \lim\limits_{x\to a} \dfrac{a-x}{ax(x-a)} = \dfrac{0}{0}$

$\lim\limits_{x\to a} \dfrac{-(x-a)}{ax(x-a)} = \lim\limits_{x\to a} \dfrac{-1}{ax} = \dfrac{-1}{a^2}$

$\Rightarrow \boxed{f'(x) = -\dfrac{1}{x^2}}$

d) $f'(a) = \lim\limits_{x\to a} \dfrac{f(x)-f(a)}{x-a} = \lim\limits_{x\to a} \dfrac{\frac{1}{x-2}-\frac{1}{a-2}}{x-a} = \lim\limits_{x\to a} \dfrac{a-x}{(x-2)(a-2)(x-a)} = \dfrac{0}{0}$

$\lim\limits_{x\to a} \dfrac{-(x-a)}{(x-2)(a-2)(x-a)} = \lim\limits_{x\to a} \dfrac{-1}{(x-2)(a-2)} = \dfrac{-1}{(a-2)^2}$

$\Rightarrow \boxed{f'(x) = -\dfrac{1}{(x-2)^2}}$

Exercice 2

a) $a'(x) = 2x(4x-5)^3 + (x^2-3)\cdot 3\cdot(4x-5)^2\cdot 4 = 2(4x-5)^2[x(4x-5)+6(x^2-3)] = 2(4x-5)^2(10x^2-5x-18)$

b) $b'(x) = \dfrac{1(x^2-3)-(x-7)\cdot 2x}{(x^2-3)^2} = \dfrac{x^2-3-2x^2+14x}{(x^2-3)^2} = \dfrac{-x^2+14x-3}{(x^2-3)^2}$

c) $c'(x) = \dfrac{[(3-2x)-2(x-5)](4x+2)-(x-5)(3-2x)\cdot 4}{4(2x+1)^2} = \dfrac{(-4x+13)(4x+2)-(-2x^2+13x-15)\cdot 4}{4(2x+1)^2} = \dfrac{-16x^2+44x+26+8x^2-52x+60}{4(2x+1)^2} = \dfrac{-8x^2-8x+86}{4(2x+1)^2} = \dfrac{-4x^2-4x+43}{2(2x+1)^2}$

d) $d'(x) = 3 + \dfrac{3}{(3x-2)^2}$

e) $e'(x) = \dfrac{1}{3}(x^3+x+1)^{-\frac{2}{3}}(3x^2+1) = \dfrac{3x^2+1}{3\cdot\sqrt[3]{(x^3+x+1)^2}}$

f) $f'(x) = (4x^2-2x)^{\frac{3}{2}} = \dfrac{3}{2}(4x^2-2x)^{\frac{1}{2}}(8x-2) = 3\sqrt{4x^2-2x}\cdot(4x-1)$

g) $g'(x) = -\dfrac{\cos(x)}{\sin^2(x)}$

h) $h'(x) = \dfrac{1}{\cos^2(x)}\cdot\cos(x) + \tan(x)\cdot(-\sin(x)) = \dfrac{1}{\cos(x)} - \dfrac{\sin(x)}{\cos(x)}\cdot\sin(x) = \dfrac{1-\sin^2(x)}{\cos(x)}$

$= \dfrac{\cos^2(x)}{\cos(x)} = \cos(x)$

Exercice 3

$f(x) = x^3 + ax^2 + bx \qquad \Rightarrow f'(x) = 3x^2 + 2ax + b$

le point T appartient à la courbe ($f(1) = 1$) : $1 + a + b = 1 \qquad \Rightarrow a + b = 0$

la tangente au point T est horizontale ($f'(1) = 0$) : $3 + 2a + b = 0 \qquad \Rightarrow 2a + b = -3$

$\Rightarrow \boxed{a = -3 \text{ et } b = 3}$

Exercice 4

a) $f(x) = 3x^2 - 6x - 5 \quad \Rightarrow \quad f(0) = -5 \quad \Rightarrow \quad T(0; -5)$

$f'(x) = 6x - 6 \quad \Rightarrow \quad f'(0) = -6$

équation de la tangente : $y + 5 = -6(x - 0) \quad \Rightarrow \quad \boxed{y = -6x - 5}$

b) $f(x) = \dfrac{4x+7}{x+3} \quad \Rightarrow \quad f(2) = \dfrac{15}{5} = 3 \quad \Rightarrow \quad T(2; 3)$

$f'(x) = \dfrac{4x + 12 - 4x - 7}{(x+3)^2} = \dfrac{5}{(x+3)^2} \quad \Rightarrow \quad f'(2) = \dfrac{5}{25} = \dfrac{1}{5}$

équation de la tangente : $y - 3 = \dfrac{1}{5}(x - 2) \quad \Rightarrow \quad 5y - 15 = x - 2$

$\Rightarrow \quad \boxed{y = \dfrac{1}{5}x + \dfrac{13}{5}}$

c) $f(x) = \dfrac{\sqrt{x}+5}{4-\sqrt{x}} \quad \Rightarrow \quad f(4) = \dfrac{7}{2} \quad \Rightarrow \quad T\left(4; \dfrac{7}{2}\right)$

$f'(x) = \dfrac{\frac{1}{2\sqrt{x}}(4-\sqrt{x}) + (\sqrt{x}+5) \cdot \frac{1}{2\sqrt{x}}}{(4-\sqrt{x})^2} = \dfrac{9}{2\sqrt{x}(4-\sqrt{x})^2} \quad \Rightarrow \quad f'(4) = \dfrac{9}{16}$

équation de la tangente : $y - \dfrac{7}{2} = \dfrac{9}{16}(x - 4) \quad \Rightarrow \quad 16y - 56 = 9x - 36$

$\Rightarrow \quad \boxed{y = \dfrac{9}{16}x + \dfrac{5}{4}}$

d) $f(x) = x^2 + \sqrt{x} - 10 \quad \Rightarrow \quad f(4) = 8 \quad \Rightarrow \quad T(4; 8)$

$f'(x) = 2x + \dfrac{1}{2\sqrt{x}} \quad \Rightarrow \quad f'(4) = 8 + \dfrac{1}{4} = \dfrac{33}{4}$

équation de la tangente : $y - 8 = \dfrac{33}{4}(x - 4) \quad \Rightarrow \quad 4y - 32 = 33x - 132$

$\Rightarrow \quad \boxed{y = \dfrac{33}{4}x - 25}$

e) $f(x) = 2\sin(x) + \cos(3x) \quad \Rightarrow \quad f(0) = 0 + 1 = 1 \quad \Rightarrow \quad T(0; 1)$

$f'(x) = 2\cos(x) - 3\sin(3x) \quad \Rightarrow \quad f'(0) = 2 - 0 = 2$

équation de la tangente : $y - 1 = 2(x - 0) \quad \Rightarrow \quad \boxed{y = 2x + 1}$

f) $f(x) = \sin^3(4x) \quad \Rightarrow \quad f(\frac{\pi}{4}) = 0 \quad \Rightarrow \quad T(\frac{\pi}{4}; 0)$

$f'(x) = 3\sin^2(4x) \cdot 4\cos(4x) = 12\sin^2(4x) \cdot \cos(4x) \quad \Rightarrow \quad f'(\frac{\pi}{4}) = 0$

équation de la tangente : $y - 0 = 0(x - \frac{\pi}{4}) \quad \Rightarrow \quad \boxed{y = 0}$