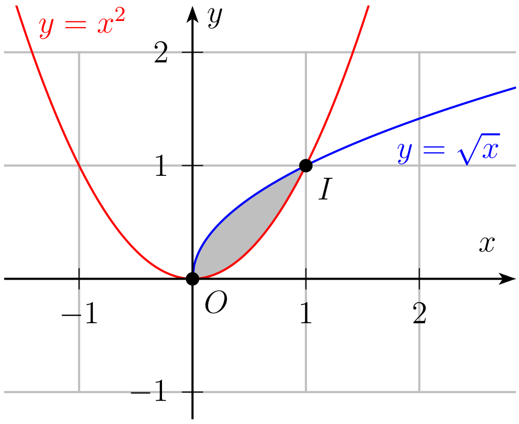
Applications de l'intégrale

Exercice 1

a)



points d'intersection : $x^2 = \sqrt{x}$

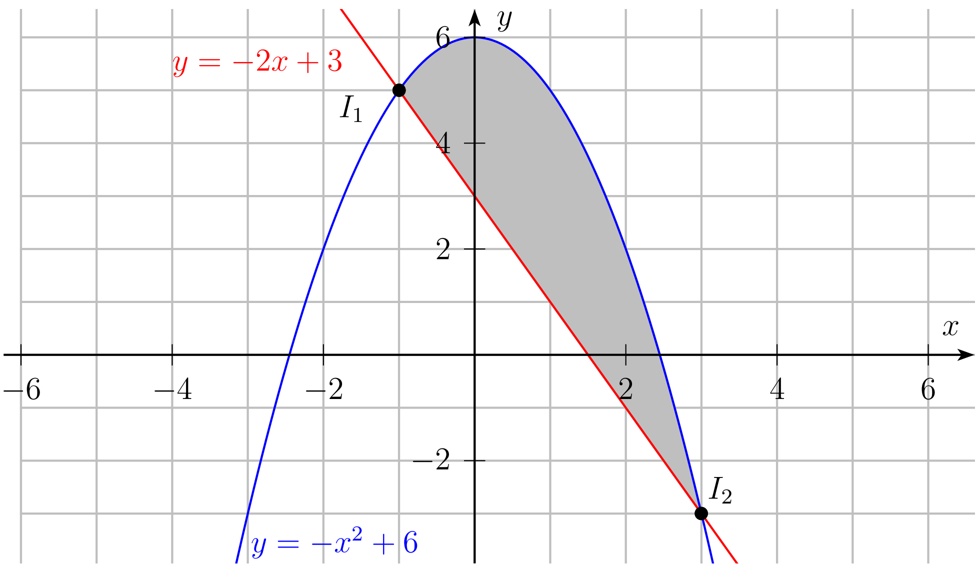
$\Rightarrow x^4 = x \qquad \Rightarrow x^4 - x = 0 \qquad \Rightarrow x(x^3-1) = 0$

$\Rightarrow x(x-1)(x^2+x+1) = 0 \qquad \Rightarrow x = 0 \quad \text{ou} \quad x = 1$

$\Rightarrow$ les points d'intersection sont $O(0;0)$ et $I(1;1)$

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right]_0^1 = \frac{2}{3} - \frac{1}{3} - 0 = \boxed{\frac{1}{3}\ u^2}$$

b)



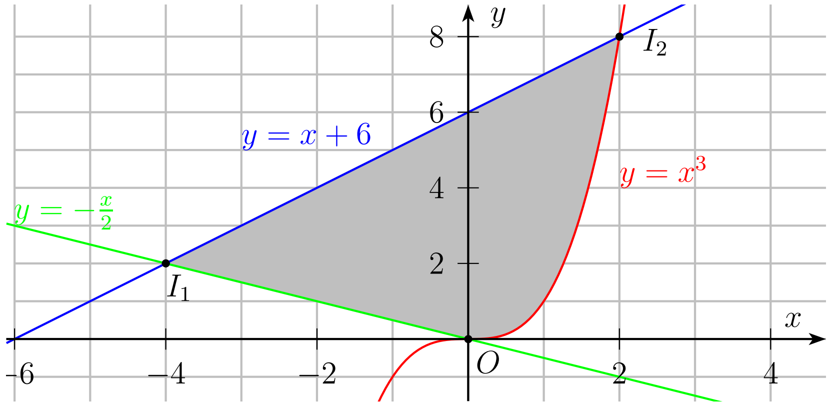
points d'intersection : $-x^2 + 6 = -2x + 3 \qquad \Rightarrow x^2 - 2x - 3 = 0$

$\Rightarrow (x-3)(x+1) = 0 \qquad \Rightarrow x = 3 \quad \text{ou} \quad x = -1$

$\Rightarrow$ les points d'intersection sont $I_1(-1;5)$ et $I_2(3;-3)$

$$A = \int_{-1}^3 (-x^2 + 2x + 3)\,dx = \left[-\frac{1}{3}x^3 + x^2 + 3x\right]_{-1}^3 = 9 + \frac{5}{3} = \boxed{\frac{32}{3}\ u^2}$$

c)



points d'intersection : $x + 6 = -\frac{x}{2}$ $\Rightarrow x = -4$ $\Rightarrow I_1(-4; 2)$

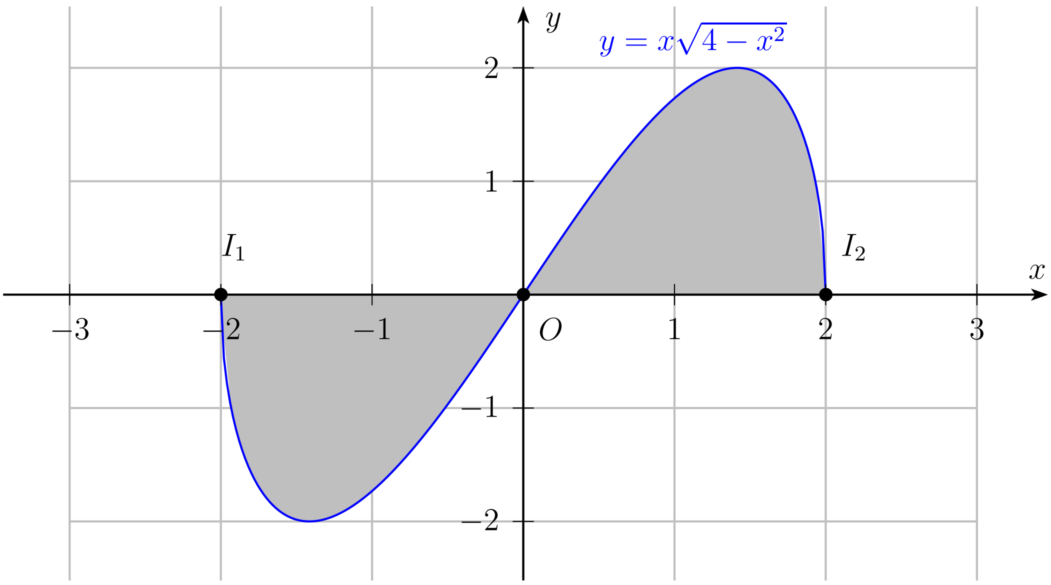
$x^3 = -\dfrac{x}{2}$ $\Rightarrow 2x^3 + x = 0$ $\Rightarrow x(2x^2 + 1) = 0$ $\Rightarrow x = 0$ $\Rightarrow O(0; 0)$

$x^3 = x + 6$ $\Rightarrow x^3 - x - 6 = 0$ $\Rightarrow (x - 2)(x^2 + 2x + 3) = 0$ $\Rightarrow x = 2$ $\Rightarrow I_2(2; 8)$

$$A = \int_{-4}^{0} (\tfrac{3}{2}x + 6)\, dx + \int_{0}^{2} (x + 6 - x^3)\, dx = \left[\frac{3}{4}x^2 + 6x\right]_{-4}^{0} + \left[\frac{1}{2}x^2 + 6x - \frac{1}{4}x^4\right]_{0}^{2}$$

$$= 0 + 12 + 10 - 0 = \boxed{22\ u^2}$$

d)

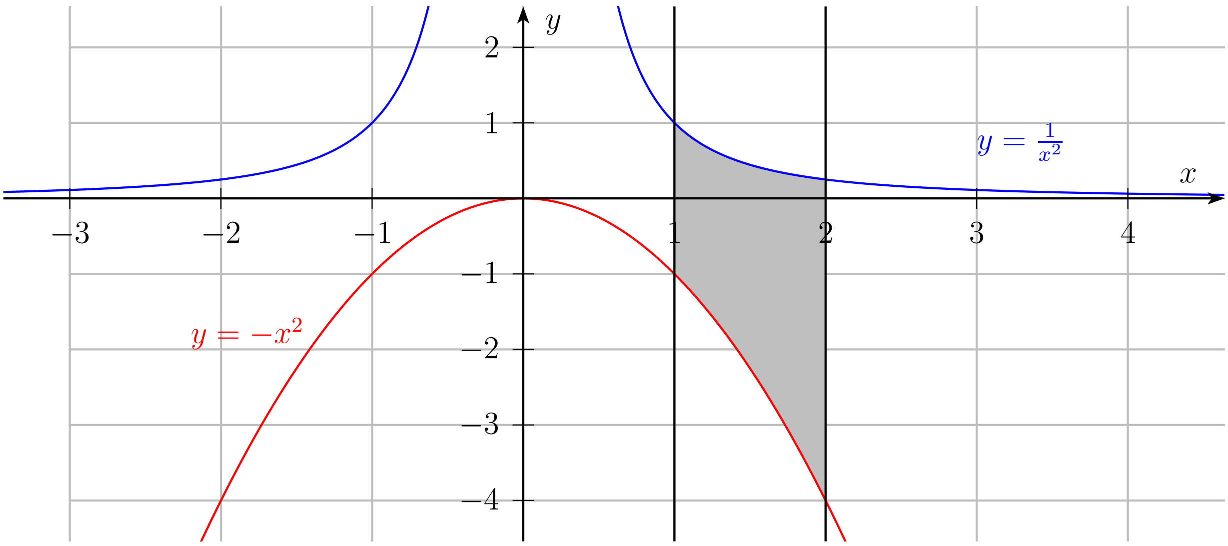


points d'intersection avec l'axe Ox : $x\sqrt{4 - x^2} = 0$ $\Rightarrow x\sqrt{(2 - x)(2 + x)} = 0$

$\Rightarrow$ les points d'intersection sont $I_1(-2; 0)$, $O(0; 0)$ et $I_2(2; 0)$

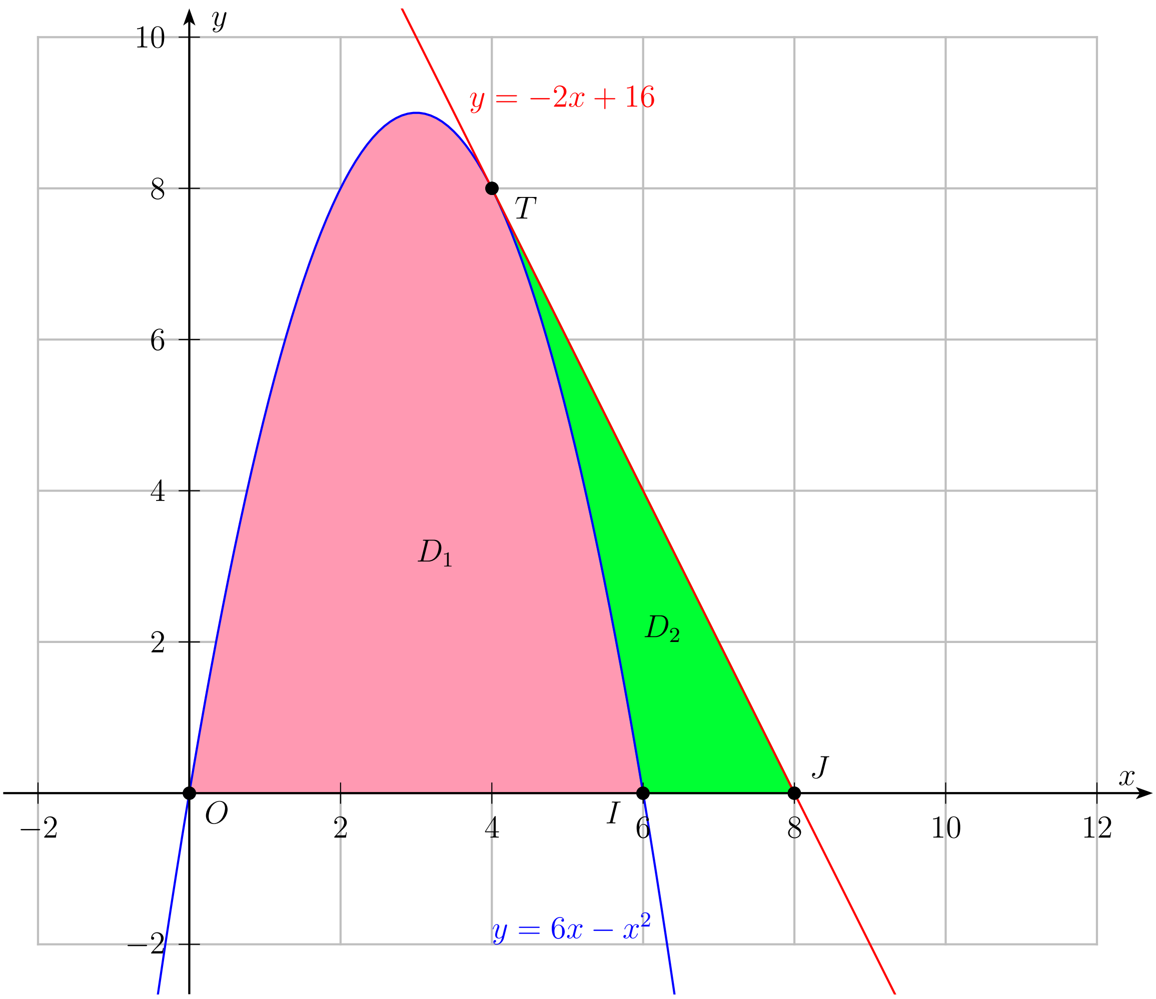
$$A = 2 \cdot \int_{0}^{2} x\sqrt{4 - x^2}\, dx = -\int_{0}^{2} \underbrace{-2x}_{u'(x)} \underbrace{\sqrt{4 - x^2}}_{u^{\frac{1}{2}}(x)}\, dx = \left[-\frac{2}{3}(4 - x^2)^{\frac{3}{2}}\right]_{0}^{2} = 0 + \frac{16}{3} = \boxed{\frac{16}{3}\ u^2}$$

e)



$$A = \int_1^2 \left(\frac{1}{x^2} + x^2 \right)\, dx = \left[-x^{-1} + \frac{1}{3}x^3 \right]_1^2 = -\frac{1}{2} + \frac{8}{3} + 1 - \frac{1}{3} = \boxed{\frac{17}{6}\ u^2}$$

Exercice 2



a) $f'(x) = 6 - 2x \qquad \Rightarrow f'(4) = 6 - 8 = -2$ (pente de la tangente au point T)

équation de la tangente t à c au point T : $\quad y = -2x + h$

$T \in t \qquad \Rightarrow 8 = -8 + h \qquad \Rightarrow h = 16$

$\Rightarrow \boxed{(t) : y = -2x + 16}$

b) Pour D_1, intersection de c avec l'axe Ox : $\quad 6x - x^2 = 0 \qquad \Rightarrow x(6-x) = 0$

$\Rightarrow x = 0 \quad$ ou $\quad x = 6 \qquad \Rightarrow O(0;0)$ et $I(6;0)$

$$A_1 = \int_0^6 (6x - x^2)\ dx = \left[3x^2 - \frac{1}{3}x^3\right]_0^6 = 36 - 0 = \boxed{36\ u^2}$$

Pour D_2, intersection de c avec l'axe d : c'est le point $T(4;8)$

intersection de d avec l'axe Ox : $\quad -2x + 16 = 0 \qquad \Rightarrow x = 8 \quad \Rightarrow J(8;0)$

$$A_2 = \int_4^6 (x^2 - 8x + 16)\ dx + \int_6^8 (-2x + 16)\ dx = \left[\frac{1}{3}x^3 - 4x^2 + 16x\right]_4^6 + \left[-x^2 + 16x\right]_6^8$$

$$= 24 - \frac{64}{3} + 64 - 60 = \boxed{\frac{20}{3}\ u^2}$$

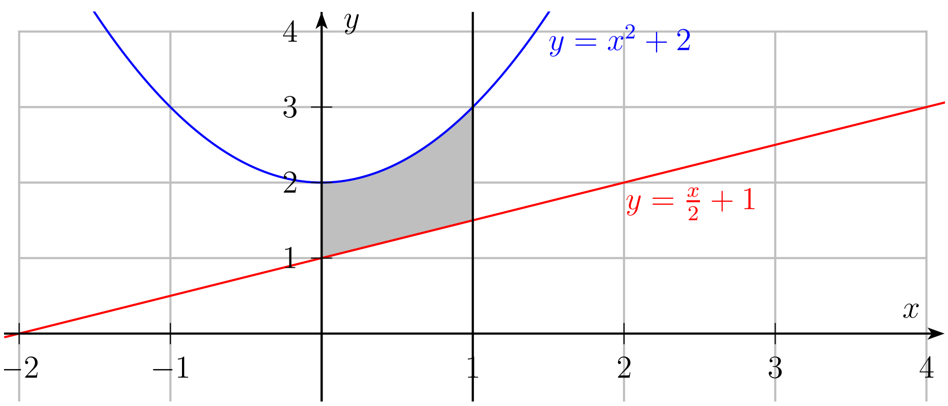
c) $$V = \pi \int_4^6 (-2x + 16)^2\ dx - \pi \int_4^6 (6x - x^2)^2\ dx + \pi \int_6^8 (-2x + 16)^2\ dx$$

$$= \pi \int_4^6 (4x^2 - 64x + 256)\ dx - \pi \int_4^6 (36x^2 - 12x^3 + x^4)\ dx + \pi \int_6^8 (4x^2 - 64x + 256)\ dx$$

$$= \pi \left[\frac{4}{3}x^3 - 32x^2 + 256x\right]_4^6 - \pi \left[12x^3 - 3x^4 + \frac{1}{5}x^5\right]_4^6 + \pi \left[\frac{4}{3}x^3 - 32x^2 + 256x\right]_6^8$$

$$= 672\pi - \frac{256\pi}{3} - 512\pi + 1296\pi - \frac{7776\pi}{5} + \frac{1024\pi}{5} + \frac{2048\pi}{3} - 672\pi = \boxed{\frac{464\pi}{15}\ u^3}$$

Exercice 3



$$V=\pi\int_0^1 \left(x^2+2\right)^2\,dx-\pi\int_0^1\left(\frac{1}{2}x+1\right)^2\,dx=\pi\int_0^1\left(x^4+4x^2+4\right)\,dx-\pi\int_0^1\left(\frac{1}{4}x^2+x+1\right)\,dx$$

$$=\pi\left[\frac{1}{5}x^5+\frac{4}{3}x^3+4x\right]_0^1-\pi\left[\frac{1}{12}x^3+\frac{1}{2}x^2+x\right]_0^1=\frac{83\pi}{15}-0-\frac{19\pi}{12}+0=\boxed{\frac{79\pi}{20}\ u^3}$$

Exercice 4

$f'(x)=5\cdot\frac{3}{2}x^{\frac{1}{2}}=\frac{15}{2}\sqrt{x}$

a) $$L=\int_0^1\sqrt{1+\frac{225}{4}x}\,dx=\frac{4}{225}\int_0^1\underbrace{\frac{225}{4}}_{u'(x)}\cdot\underbrace{\sqrt{1+\frac{225}{4}x}}_{u^{\frac{1}{2}}(x)}\,dx=\left[\frac{4}{225}\cdot\frac{2}{3}\cdot\left(1+\frac{225}{4}x\right)^{\frac{3}{2}}\right]_0^1=$$

$$\left[\frac{8}{675}\cdot\sqrt{\left(1+\frac{225}{4}x\right)^3}\right]_0^1=\frac{229\sqrt{229}}{675}-\frac{8}{675}=\boxed{\frac{1}{675}\left(229\sqrt{229}-8\right)\ u}$$

b) $$V=\pi\int_0^1 25x^3\,dx=\pi\left[\frac{25}{4}x^4\right]_0^1=\frac{25\pi}{4}-0=\boxed{\frac{25\pi}{4}\ u^3}$$

Exercice 5

$f'(x)=3\cdot\frac{x^2}{3}-\frac{4}{16x^2}=x^2-\frac{1}{4x^2}$

$$A=2\pi\int_1^2\left(\frac{4x^4+3}{12x}\right)\sqrt{1+x^4-\frac{1}{2}+\frac{1}{16x^4}}\,dx=2\pi\int_1^2\left(\frac{4x^4+3}{12x}\right)\sqrt{x^4+\frac{1}{2}+\frac{1}{16x^4}}\,dx=$$

$$2\pi\int_1^2\left(\frac{4x^4+3}{12x}\right)\left(x^2+\frac{1}{4x^2}\right)\,dx=2\pi\int_1^2\left(\frac{4x^4+3}{12x}\right)\left(\frac{4x^4+1}{4x^2}\right)\,dx=2\pi\int_1^2\left(\frac{16x^8+16x^4+3}{48x^3}\right)\,dx$$

$$=2\pi\int_1^2\left(\frac{1}{3}x^5+\frac{1}{3}x+\frac{1}{16x^3}\right)\,dx=2\pi\left[\frac{1}{18}x^6+\frac{1}{6}x^2-\frac{1}{32x^2}\right]_1^2=\frac{4855\pi}{576}-\frac{55\pi}{144}=\boxed{\frac{515\pi}{64}\ u^2}$$