Fonctions logarithmiques et exponentielles

Exercice 1

a) $\lim\limits_{x\to 2} \dfrac{e^x - e^2}{x-2} = \dfrac{0}{0}$ (f.i.) $\quad \Rightarrow$ (L'Hospital) $\lim\limits_{x\to 2} \dfrac{e^x}{1} = \boxed{e^2}$

b) $\lim\limits_{x\to 0} \dfrac{x\cdot e^x}{1-e^x} = \dfrac{0}{0}$ (f.i.) $\quad \Rightarrow$ (L'Hospital) $\lim\limits_{x\to 0} \dfrac{e^x + x\cdot e^x}{-e^x} = \boxed{-1}$

c) $\lim\limits_{x\to -1} \dfrac{\ln(2+x)}{x+1} = \dfrac{0}{0}$ (f.i.) $\quad \Rightarrow$ (L'Hospital) $\lim\limits_{x\to -1} \dfrac{\frac{1}{2+x}}{1} = \boxed{1}$

d) $\lim\limits_{x\to -\infty} \dfrac{e^{-x^2}}{x^2} = \dfrac{0}{+\infty} = \boxed{0}$

e) $\lim\limits_{x\to +\infty} \dfrac{e^x}{x^2+2x+3} = \dfrac{+\infty}{+\infty}$ (f.i.) $\quad \Rightarrow$ (L'Hospital) $\lim\limits_{x\to +\infty} \dfrac{e^x}{2x+2} = \dfrac{+\infty}{+\infty}$ (f.i.)

$\Rightarrow$ (Hospital) $\lim\limits_{x\to +\infty} \dfrac{e^x}{2} = \dfrac{+\infty}{2} = \boxed{+\infty}$

f) $\lim\limits_{x\to 0} \dfrac{\ln(\cos(x))}{x^2} = \dfrac{0}{0}$ (f.i.) $\quad \Rightarrow$ (L'Hospital) $\lim\limits_{x\to 0} \dfrac{\frac{-\sin(x)}{\cos(x)}}{2x} = \dfrac{0}{0}$ (f.i.)

$\Rightarrow$ (Hospital) $\lim\limits_{x\to 0} \dfrac{\frac{-\cos^2(x)-\sin^2(x)}{\cos^2(x)}}{2} = \lim\limits_{x\to 0} \dfrac{\frac{-1}{\cos^2(x)}}{2} = \boxed{-\dfrac{1}{2}}$

Exercice 2

a) $f(x) = \ln(3x^2-2x)$

$ED(f)$: $3x^2 - 2x > 0 \quad \Rightarrow x(3x-2) > 0 \quad \Rightarrow ED(f) =]-\infty; 0[\cup]\frac{2}{3}; +\infty[$

zéros de f: $3x^2 - 2x = 1 \quad \Rightarrow 3x^2 - 2x - 1 = 0 \quad \Rightarrow (3x+1)(x-1) = 0$

$\Rightarrow x = 1 \quad$ et $\quad x = -\frac{1}{3}$

x	$-\infty$		$-\frac{1}{3}$		0		$\frac{2}{3}$		1		$+\infty$
$f(x)$		$+$	0	$-$	‖	//////	‖	$-$	0	$+$	

$\lim\limits_{\substack{x\to 0\\ <}} \ln(3x^2-2x) = -\infty \quad \Rightarrow$ asymptote verticale en $x = 0$ à droite de la courbe

$\lim\limits_{\substack{x\to \frac{2}{3}\\ >}} \ln(3x^2-2x) = -\infty \quad \Rightarrow$ asymptote verticale en $x = \frac{2}{3}$ à gauche de la courbe

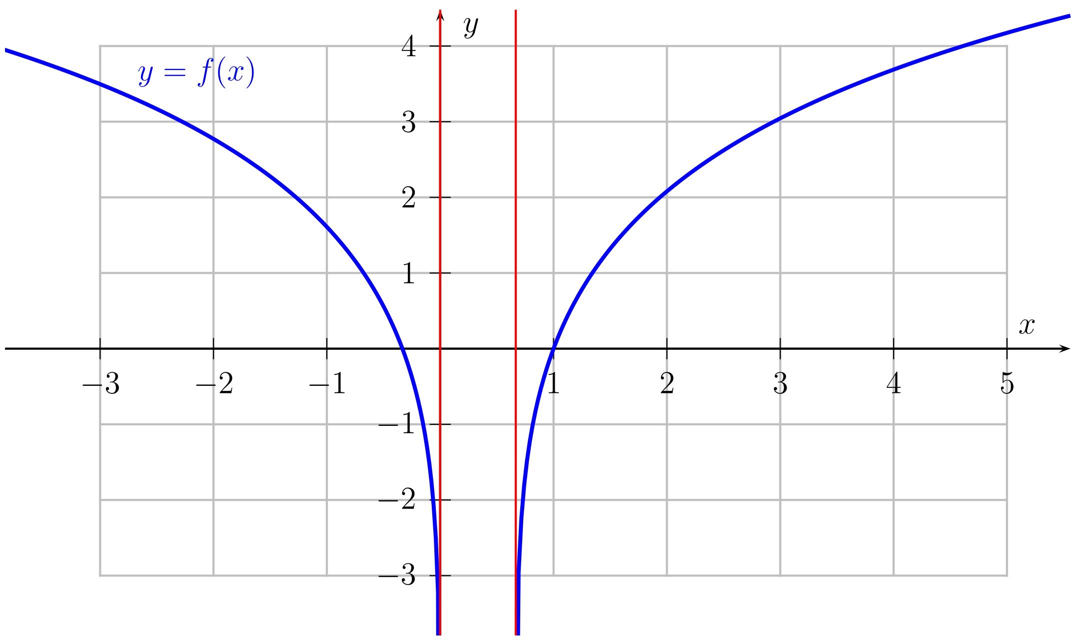
$$\left.\begin{array}{l}\lim\limits_{x\to-\infty} \ln(3x^2-2x) = +\infty \\ \lim\limits_{x\to+\infty} \ln(3x^2-2x) = +\infty\end{array}\right\} \Rightarrow \text{pas d'asymptote horizontale}$$

dérivée: $f'(x) = \dfrac{6x-2}{3x^2-2x} \quad \Rightarrow ED(f') = ED(f)$

zéros de f': $6x-2=0 \quad \Rightarrow x = \frac{1}{3} \notin ED(f') \quad \Rightarrow$ pas de zéro pour f'

x	$-\infty$		0		$\frac{1}{3}$		$\frac{2}{3}$		$+\infty$
$6x-2$		$-$		$-$	0	$+$		$+$	
$3x^2-2x$		$+$	0	$-$		$-$	0	$+$	
$f'(x)$		$-$	‖				‖	$+$	
		$\searrow$						$\nearrow$	

$\Rightarrow$ pas d'extremum



b) $f(x) = (x-2)^2 \cdot e^x \quad \Rightarrow ED(f) = \mathbb{R}$

zéros de f: $x-2=0 \quad \Rightarrow x=2$

x	$-\infty$		2		$+\infty$
$f(x)$		$+$	0	$+$	

$$\lim_{x\to+\infty} (x-2)^2 \cdot e^x = +\infty$$

$$\lim_{x\to-\infty} (x-2)^2 \cdot e^x = +\infty \cdot 0 \text{ (f.i.)} \qquad \Rightarrow \lim_{x\to-\infty} \frac{(x-2)^2}{e^{-x}} = \frac{+\infty}{+\infty} \text{ (f.i.)}$$

$$\Rightarrow \text{ (L'Hospital) } \lim_{x\to-\infty} \frac{2(x-2)}{-e^{-x}} = \frac{-\infty}{-\infty} \text{ (f.i.)}$$

$$\Rightarrow \text{ (L'Hospital) } \lim_{x\to-\infty} \frac{2}{e^{-x}} = \frac{2}{+\infty} = 0$$

$\Rightarrow$ asymptote horizontale $(x \to -\infty)$: $x = 0$

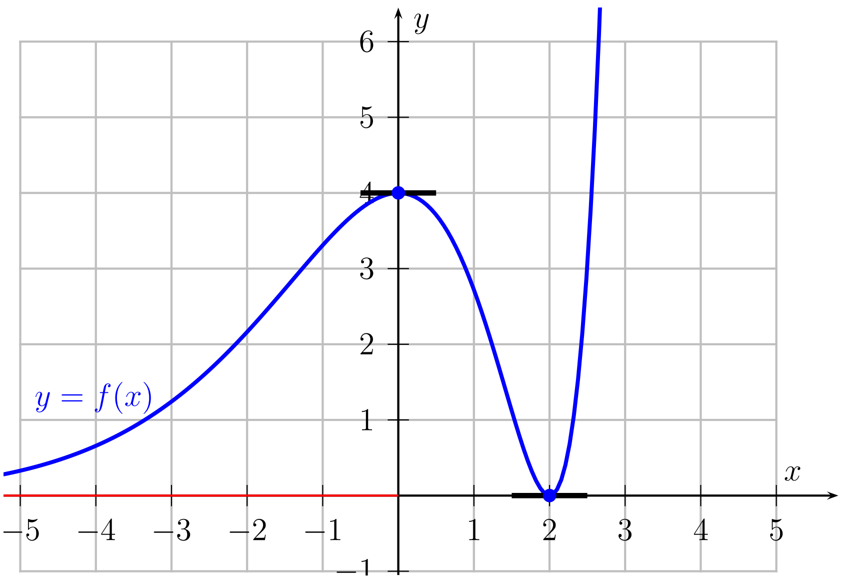
dérivée: $f'(x) = 2(x-2)\cdot e^x + (x-2)^2 \cdot e^x = e^x(2x-4+x^2-4x+4) = e^x(x^2-2x)$

$\Rightarrow ED(f') = ED(f)$

zéros de f': $x^2 - 2x = 0 \quad \Rightarrow x(x-2) = 0 \quad \Rightarrow x = 0$ et $x = 2$

x	$-\infty$		0		2		$+\infty$
e^x		$+$		$+$		$+$	
x^2-2x		$+$	0	$-$	0	$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
		$\nearrow$	Max	$\searrow$	min	$\nearrow$	

extremums: maximum $(0;4)$ et minimum $(2;0)$



Exercice 3

a) $f(x) = \ln(5x) \quad \Rightarrow ED(f) = \mathbb{R}_+^*$

$$f'(x) = \frac{5}{5x} = \boxed{\frac{1}{x}}$$

b) $f(x) = \ln\left(\frac{x^2}{1-x}\right) \quad \Rightarrow ED(f) =]-\infty; 0[\cup]0; 1[$

$$f'(x) = \frac{2x(1-x)+x^2}{(1-x)^2} \div \frac{x^2}{1-x} = \frac{2x-x^2}{(1-x)^2} \cdot \frac{1-x}{x^2} = \frac{2-x}{x(1-x)} = \boxed{\frac{x-2}{x^2-x}}$$

c) $f(x) = x \cdot \ln(x) - x \quad \Rightarrow ED(f) = \mathbb{R}_+^*$

$$f'(x) = \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x) + 1 - 1 = \boxed{\ln(x)}$$

d) $f(x) = e^{5x} \quad \Rightarrow ED(f) = \mathbb{R}$

$$f'(x) = \boxed{5 \cdot e^{5x}}$$

e) $f(x) = x^2 \cdot e^x \quad \Rightarrow ED(f) = \mathbb{R}$

$$f'(x) = 2x \cdot e^x + x^2 \cdot e^x = \boxed{e^x(x^2+2x)}$$

f) $f(x) = e^{\sin(x)} \quad \Rightarrow ED(f) = \mathbb{R}$

$$f'(x) = \boxed{\cos(x) \cdot e^{\sin(x)}}$$

Exercice 4

a) $$\int_1^4 \frac{1}{2x+3}\,dx = \frac{1}{2}\int_1^4 \frac{\overbrace{2}^{u'(x)}}{\underbrace{2x+3}_{u(x)}}\,dx = \left[\frac{1}{2}\ln|2x+3|\right]_1^4 = \frac{\ln(11)}{2} - \frac{\ln(5)}{2} = \boxed{\frac{1}{2}\ln\left(\frac{11}{5}\right) \quad u^2}$$

b) $$\int_1^2 \frac{3x^2-4x+1}{2x^3-4x^2+2x+8}\,dx = \frac{1}{2}\int_1^2 \frac{\overbrace{6x^2-8x+2}^{u'(x)}}{\underbrace{2x^3-4x^2+2x+8}_{u(x)}}\,dx = \left[\frac{1}{2}\ln|2x^3-4x^2+2x+8|\right]_1^2 =$$

$$\frac{\ln(12)}{2} - \frac{\ln(8)}{2} = \boxed{\frac{1}{2}\ln\left(\frac{3}{2}\right) \quad u^2}$$

c) $$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan(x)\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin(x)}{\cos(x)}\,dx = -\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\overbrace{-\sin(x)}^{u'(x)}}{\underbrace{\cos(x)}_{u(x)}}\,dx = [-\ln|\cos(x)|]_{\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$-\ln\left(\frac{1}{2}\right) + \ln\left(\frac{\sqrt{3}}{2}\right) = \ln(\sqrt{3}) = \boxed{\frac{1}{2}\ln(3)\ u^2}$$

d) $$\int_1^3 x^2 \cdot e^{x^3}\,dx = \frac{1}{3}\int_1^3 \underbrace{3x^2}_{u'(x)} \cdot \underbrace{e^{x^3}}_{e^{u(x)}}\,dx = \left[\frac{1}{3}e^{x^3}\right]_1^3 = \frac{e^{27}}{3} - \frac{e}{3} = \boxed{\frac{e^{27}-e}{3}\ u^2}$$

e) $$\int_{-2}^3 e^{2x+1}\,dx = \frac{1}{2}\int_{-2}^3 \underbrace{2}_{u'(x)} \cdot \underbrace{e^{2x+1}}_{e^{u(x)}}\,dx = \left[\frac{1}{2}e^{2x+1}\right]_{-2}^3 = \frac{e^7}{2} - \frac{e^{-3}}{2} = \boxed{\frac{e^{10}-1}{2e^3}\ u^2}$$

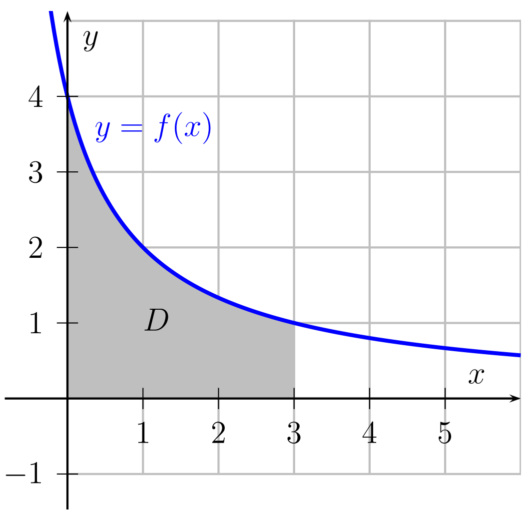
f) $$\int_1^4 \frac{1}{\sqrt{x}\cdot e^{\sqrt{x}}}\,dx = \int_1^4 x^{-\frac{1}{2}} \cdot e^{-x^{\frac{1}{2}}}\,dx = (-2)\int_1^4 \underbrace{-\frac{1}{2}x^{-\frac{1}{2}}}_{u'(x)} \cdot \underbrace{e^{-x^{\frac{1}{2}}}}_{e^{u(x)}}\,dx = \left[(-2)\cdot e^{-x^{\frac{1}{2}}}\right]_1^4$$

$$= -2e^{-2} + 2e^{-1} = \frac{2}{e} - \frac{2}{e^2} = \boxed{\frac{2(e-1)}{e^2}\ u^2}$$

Exercice 5

$$\ln\left(e^4\right) + 2\cdot\ln\left(\sqrt{e^5}\right) - \ln\left(e^{\frac{1}{4}}\right) = 4 + 2\cdot\tfrac{5}{2} - \frac{1}{4} = 4 + 5 - \frac{1}{4} = \boxed{\frac{35}{4}}$$

Exercice 6



a) $$A = \int_0^3 \frac{4}{x+1}\,dx = 4\int_0^3 \frac{\overbrace{1}^{u'(x)}}{\underbrace{x+1}_{u(x)}} = [4\cdot\ln|x+1|]_0^3 = 4\cdot\ln(4) - 0 = \boxed{4\cdot\ln(4)\ u^2}$$

b) $V = \pi \int_0^3 \frac{16}{(x+1)^2}\, dx = 16\pi \int_0^3 \underbrace{1}_{u'(x)} \underbrace{(x+1)^{-2}}_{u^{-2}(x)} = [16\pi \cdot (-1) \cdot (x+1)^{-1}]_0^3 = \left[-\frac{16\pi}{x+1}\right]_0^3$

$= -4\pi + 16\pi = \boxed{12\pi\ u^3}$

Exercice 7

a) $-2x^2 + 3x = 0 \quad \Rightarrow x(-2x+3) = 0 \quad \Rightarrow \boxed{x = 0 \quad \text{ou} \quad x = \frac{3}{2}}$

b) $ED(f) = \mathbb{R}$

$f'(x) = (-4x+3) \cdot e^{-x} + (-2x^2+3x) \cdot (-1) \cdot e^{-x} = e^{-x} \cdot (2x^2 - 7x + 3) \quad ED(f') = \mathbb{R}$

zéros de f': $2x^2 - 7x + 3 = 0 \quad \Rightarrow (2x-1)(x-3) = 0 \quad \Rightarrow x = \frac{1}{2} \quad$ ou $\quad x = 3$

x	$-\infty$		$\frac{1}{2}$		3		$+\infty$
e^{-x}		$+$		$+$		$+$	
$2x^2 - 7x + 3$		$+$	0	$-$	0	$+$	
$f'(x)$		$+$	0	$-$	0	$+$	
		$\nearrow$	Max	$\searrow$	min	$\nearrow$	

$\boxed{\text{extremums: maximum } (0,5; 0,61) \text{ et minimum } (3; -0,45)}$

Exercice 8

a) $P(0) = \boxed{80 \text{ personnes}}$

b) $P(2) \cong \boxed{152 \text{ personnes}}$

c) $P(8) \cong \boxed{184 \text{ personnes}}$

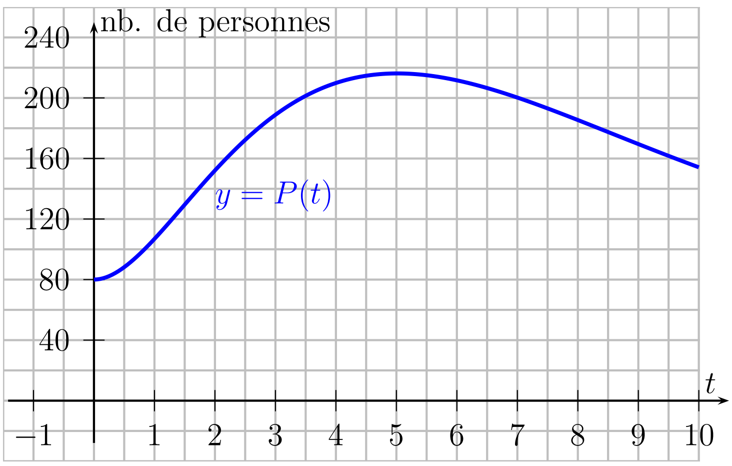
d) $P'(t) = 80t \cdot e^{-0,4t} + 40t^2 \cdot (-0,4) \cdot e^{-0,4t} = e^{-0,4t}(-16t^2 + 80t)$

zéros de P': $-16t^2 + 80t = 0 \quad \Rightarrow 16t(-t+5) = 0 \quad \Rightarrow t = 0 \quad$ ou $\quad t = 5$

t	$-\infty$		0		5		$+\infty$
$e^{-0,4t}$		$+$		$+$		$+$	
$-16t^2+80t$		$-$	0	$+$	0	$-$	
$P'(t)$		(hachuré)	0	$+$	0	$-$	
				$\nearrow$	Max	$\searrow$	

maximum $(5; \sim 215)$ $\Rightarrow$ **215 personnes au maximum atteintes**

e)



Exercice 9

a) $\boxed{ED(f) = \mathbb{R}^*}$

b) $f'(x) = 2 - \ln(x^2) - x \cdot \dfrac{2x}{x^2} = 2 - \ln(x^2) - 2 = -\ln(x^2) \quad ED(f') = ED(f)$

zéros de f': $x^2 = 1 \quad \Rightarrow x = -1 \quad \text{ou} \quad x = 1$

x	$-\infty$		-1		0		1		$+\infty$
$f'(x)$		$-$	0	$+$	$\Vert$	$+$	0	$-$	
		$\searrow$	min	$\nearrow$		$\nearrow$	Max	$\searrow$	

extremums: maximum $(1; 2)$ et minimum $(-1; -2)$

Exercice 10

$$V = \pi \int_0^{\ln(3)} e^{-6x}\, dx = -\frac{\pi}{6} \int_0^{\ln(3)} \underbrace{-6}_{u'(x)}\, \underbrace{e^{-6x}}_{e^{u(x)}} = \left[-\frac{\pi}{6} \cdot e^{-6x} \right]_0^{\ln(3)} = -\frac{\pi}{4374} + \frac{\pi}{6} = \boxed{\frac{364\pi}{2187}\ u^3}$$