**2.5.5**

a) $\dfrac{3}{n-2} < 0.1 \quad \Rightarrow \quad 30 < n-2 \quad \Rightarrow \quad n > 32 \quad \Rightarrow \quad N = 33$

$$\frac{3}{n-2} < \epsilon \quad \Rightarrow \quad \frac{3}{\epsilon} < n-2 \quad \Rightarrow n > \frac{3}{\epsilon} + 2 = \frac{3+2\epsilon}{\epsilon}$$

$$\Rightarrow \quad n_0 = E\left(\frac{3+2\epsilon}{\epsilon} + 1\right) = E\left(\frac{3(1+\epsilon)}{\epsilon}\right)$$

$$\Rightarrow \quad \forall n \geq n_0 \qquad |u_n - 0| < \epsilon \quad \Rightarrow \quad L = \lim_{n\to\infty} u_n = 0$$

b) $\left|\dfrac{n}{n+1} - 1\right| < 0.25 \quad \Rightarrow \quad \left|\dfrac{-1}{n+1}\right| < 0.25 \quad \Rightarrow \quad \dfrac{1}{n+1} < 0.25$

$$\Rightarrow \quad 4 < n+1 \quad \Rightarrow \quad n > 3 \quad \Rightarrow \quad N = 4$$

$$\left|\frac{n}{n+1} - 1\right| < \epsilon \quad \Rightarrow \quad \left|\frac{-1}{n+1}\right| < \epsilon \quad \Rightarrow \quad \frac{1}{n+1} < \epsilon$$

$$\Rightarrow \quad \frac{1}{\epsilon} < n+1 \quad \Rightarrow \quad n > \frac{1}{\epsilon} - 1 = \frac{1-\epsilon}{\epsilon}$$

$$\Rightarrow \quad n_0 = E\left(\frac{1-\epsilon}{\epsilon} + 1\right) = E\left(\frac{1}{\epsilon}\right)$$

$$\Rightarrow \quad \forall n \geq n_0 \qquad |u_n - 1| < \epsilon \quad \Rightarrow \quad L = \lim_{n\to\infty} u_n = 1$$

**2.5.6**

a) $\left|\dfrac{4n}{2n-1} - 2\right| = \left|\dfrac{2}{2n-1}\right| = \dfrac{2}{2n-1} < \epsilon \quad \Rightarrow \quad 2n-1 > \dfrac{2}{\epsilon} \quad \Rightarrow \quad 2n > \dfrac{2}{\epsilon} + 1$

$$\Rightarrow \quad n > \frac{1}{\epsilon} + \frac{1}{2} \quad \Rightarrow \quad n_0 = E\left(\frac{1}{\epsilon} + \frac{1}{2} + 1\right) = E\left(\frac{1}{\epsilon} + \frac{3}{2}\right)$$

$$\Rightarrow \quad \forall n \geq n_0 \qquad |u_n - 2| < \epsilon \quad \Rightarrow \quad L = \lim_{n\to\infty} u_n = 2$$

b) $\left|\dfrac{2n^2}{1-n^2} + 2\right| = \left|\dfrac{2}{1-n^2}\right| = \dfrac{2}{n^2-1} < \epsilon \quad \Rightarrow \quad n^2 - 1 > \dfrac{2}{\epsilon} \quad \Rightarrow \quad n^2 > \dfrac{2}{\epsilon} + 1$

$$\Rightarrow \quad n > \sqrt{\frac{2}{\epsilon} + 1} \quad \Rightarrow \quad n_0 = E\left(\sqrt{\frac{2}{\epsilon} + 1} + 1\right)$$

$$\Rightarrow \quad \forall n \geq n_0 \qquad |u_n - (-2)| < \epsilon \quad \Rightarrow \quad L = \lim_{n\to\infty} u_n = -2$$

c) $\left|\dfrac{5}{n^3} - 0\right| = \dfrac{5}{n^3} < \epsilon \quad \Rightarrow \quad n^3 > \dfrac{5}{\epsilon} \quad \Rightarrow \quad n > \sqrt[3]{\dfrac{5}{\epsilon}} \quad \Rightarrow \quad n_0 = E\left(\sqrt[3]{\dfrac{5}{\epsilon}} + 1\right)$

$$\Rightarrow \quad \forall n \geq n_0 \qquad |u_n - 0| < \epsilon \quad \Rightarrow \quad L = \lim_{n\to\infty} u_n = 0$$

d) pas de limite (si $n$ est pair $\to 5$, si $n$ est impair $\to 1$)

e) $$\frac{(2n-1)^4}{(1-n^2)^2} = \frac{16n^4 - 32n^3 + 24n^2 - 8n + 1}{1 - 2n^2 + n^4} = \frac{16 - \frac{32}{n} + \frac{24}{n^2} - \frac{8}{n^3} + \frac{1}{n^4}}{\frac{1}{n^4} - \frac{2}{n^2} + 1}$$

$$\lim_{n\to\infty} \frac{32}{n} = \lim_{n\to\infty} \frac{24}{n^2} = \lim_{n\to\infty} \frac{8}{n^3} = \lim_{n\to\infty} \frac{1}{n^4} = \lim_{n\to\infty} \frac{2}{n^2} = 0$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{(2n-1)^4}{(1-n^2)^2} = \frac{16}{1} = 16$$

f) $$\frac{7n^2}{\sqrt{n^4-5}} = \frac{7}{\frac{\sqrt{n^4-5}}{n^2}} = \frac{7}{\sqrt{\frac{n^4-5}{n^4}}} = \frac{7}{\sqrt{1 - \frac{5}{n^4}}}$$

$$\lim_{n\to\infty} \frac{5}{n^4} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{7n^2}{\sqrt{n^4-5}} = \frac{7}{\sqrt{1}} = 7$$

**2.5.7**

a) $$-1 \leq \cos(n) \leq 1 \quad \Rightarrow \quad -\frac{1}{n^2} \leq \frac{\cos(n)}{n^2} \leq \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{-1}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{\cos(n)}{n^2} = 0$$

b) $$-1 \leq (-1)^n \leq 1 \quad \Rightarrow \quad -\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}$$

$$\lim_{n\to\infty} \frac{-1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

c) $$-n \leq n\sin(n!) \leq n \quad \Rightarrow \quad -\frac{n}{n^2+1} \leq \frac{n\sin(n!)}{n^2+1} \leq \frac{n}{n^2+1}$$

$$\lim_{n\to\infty} \frac{-n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{-1}{n}}{1+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n}{n^2+1} = 0 \quad \Rightarrow \lim_{n\to\infty} \frac{n\sin(n!)}{n^2+1} = 0$$

d) $$1 \leq n! \leq n^{n-1} \quad \Rightarrow \quad \frac{1}{n^n} \leq \frac{n!}{n^n} \leq \frac{n^{n-1}}{n^n}$$

$$\lim_{n\to\infty} \frac{1}{n^n} = \lim_{n\to\infty} \frac{n^{n-1}}{n^n} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n!}{n^n} = 0$$

e) $$0 \leq \sin\left(\tfrac{1}{n^2}\right) \leq \frac{1}{n^2} \quad \Rightarrow \quad 0 \leq n\sin\left(\frac{1}{n^2}\right) \leq n \cdot \frac{1}{n^2} = \frac{1}{n}$$

$$\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n\to\infty} n\sin\left(\frac{1}{n^2}\right) = 0$$

f) $$nx - 1 \leq E(nx) \leq nx \quad \Rightarrow \quad \frac{nx-1}{n} \leq \frac{1}{n}E(nx) \leq \frac{nx}{n}$$

$$\lim_{n\to\infty} \frac{nx-1}{n} = \lim_{n\to\infty} x - \frac{1}{n} = \lim_{n\to\infty} \frac{nx}{n} = x \quad \Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}E(nx) = x$$

### 2.5.8

a) $P(n) : u_n < 2 \qquad u_1 = \sqrt{2} < 2 \quad \Rightarrow \quad P(1)$ est vraie

supposons que $P(n)$ est vraie pour un entier $n \geq 1$

$\Rightarrow \quad u_n < \sqrt{2} \quad \Rightarrow \quad u_{n+1} = \sqrt{2u_n} < \sqrt{2 \cdot 2} = 2 \quad \Rightarrow \quad P(n+1)$ est vraie

$\Rightarrow \quad (u_n)_{n\geq 1}$ majorée par 2

$u_{n+1} - u_n =$

$$\sqrt{2u_n} - u_n = \frac{(\sqrt{2u_n} - u_n)(\sqrt{2u_n} + u_n)}{\sqrt{2u_n} + u_n} = \frac{2u_n - u_n^2}{\sqrt{2u_n} + u_n} = \frac{u_n \overbrace{(2 - u_n)}^{>0}}{\sqrt{2u_n} + u_n} > 0$$

$\Rightarrow \quad (u_n)_{n\geq 1}$ est croissante

b) $(u_n)_{n\geq 1}$ suite croissante et majorée $\Rightarrow$ suite convergente

$L = \sqrt{2L} \quad \Rightarrow \quad L^2 = 2L \quad \Rightarrow \quad L^2 - 2L = 0 \quad \Rightarrow \quad L(L-2) = 0$

comme $(u_n)_{n\geq 1} > \sqrt{2} \quad \Rightarrow \quad \lim\limits_{n\to\infty} u_n = 2$

### 2.5.9

a) $P(n) : u_n > 0 \qquad u_1 = 2 > 0 \quad \Rightarrow \quad P(1)$ est vraie

supposons que $P(n)$ est vraie pour un entier $n \geq 1$

$$\Rightarrow \quad u_n > 0 \quad \Rightarrow \quad u_{n+1} = \frac{u_n}{u_n + 1} > \frac{0}{\underbrace{u_n + 1}_{\neq 0}} = 0 \quad \Rightarrow \quad P(n+1) \text{ est vraie}$$

$\Rightarrow \quad (u_n)_{n\geq 1}$ minorée par 0

$$u_{n+1} - u_n = \frac{u_n}{u_n + 1} - u_n = \frac{u_n - u_n^2 - u_n}{u_n + 1} = -\frac{u_n^2}{u_n + 1} < 0$$

$\Rightarrow \quad (u_n)_{n\geq 1}$ est décroissante

b) $(u_n)_{n\geq 1}$ suite décroissante et minorée $\Rightarrow$ suite convergente

$$L = \frac{L}{L+1} \quad \Rightarrow \quad L^2 + L = L \quad \Rightarrow \quad L^2 = 0 \quad \Rightarrow \quad L = 0$$

$\Rightarrow \quad \lim\limits_{n\to\infty} u_n = 0$