Nombres complexes

Exercice 1.

a) $z = \sqrt{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) = 1 - i$

$z \cdot w = (1-i)(5+3i) = \boxed{8-2i}$

b) $\frac{z}{w} = \frac{1-i}{5+3i} \cdot \frac{5-3i}{5-3i} = \frac{2-8i}{34} = \boxed{\frac{1}{17} - \frac{4}{17}i}$

c) $z^4 = 4 \text{ cis } 7\pi = 4(\cos(\pi) + i\sin(\pi)) = \boxed{-4}$

d) $z \cdot \overline{z} = (1-i)(1+i) = \boxed{2}$

Exercice 2.

$|z| = \sqrt{1+3} = 2 \qquad \cos(\theta_z) = -\frac{1}{2} \qquad \sin(\theta_z) = -\frac{\sqrt{3}}{2} \quad \Rightarrow \quad \theta_z = \frac{4\pi}{3}$

$\Rightarrow \quad z = \boxed{2 \text{ cis } \frac{4\pi}{3}}$

$|w| = 6 \qquad \cos(\theta_w) = 0 \qquad \sin(\theta_w) = -1 \quad \Rightarrow \quad \theta_w = \frac{3\pi}{2}$

$\Rightarrow \quad w = \boxed{6 \text{ cis } \frac{3\pi}{2}}$

Exercice 3.

$\Delta = 4(1+i)^2 + 20(1+2i) = 8i + 20 + 40i = 20 + 48i$

$$\begin{cases} a^2 + b^2 = 52 \\ a^2 - b^2 = 20 \\ ab = 24 \end{cases} \quad \Rightarrow \quad 2a^2 = 72 \quad \Rightarrow \quad a^2 = 36$$

$\Rightarrow \quad a = \pm 6 \quad \Rightarrow \quad b = \pm 4 \quad \Rightarrow \quad \sqrt{\Delta} = \pm(6+4i) \quad \Rightarrow \quad z = \frac{-2-2i \pm (6+4i)}{2}$

$\Rightarrow \quad z_1 = 2+i \qquad z_2 = -4-3i \quad \Rightarrow \boxed{\mathcal{S} = \{2+i; -4-3i\}}$

Exercice 4.

a) $|z^6| = 1 \qquad \cos(\theta) = -1 \qquad \sin(\theta) = 0 \quad \Rightarrow \quad \theta = \pi$

$$\Rightarrow \quad \begin{cases} s = \sqrt[6]{1} \\ 6\alpha = \pi + k \cdot 2\pi \end{cases} \quad \Rightarrow \quad \begin{cases} s = 1 \\ \alpha = \frac{\pi}{6} + k \cdot \frac{\pi}{3} \quad k = 0, \ldots, 5 \end{cases}$$

$w_1 = 1 \text{ cis } \frac{\pi}{6} = \cos(\frac{\pi}{6}) + i \sin(\frac{\pi}{6}) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_2 = 1 \text{ cis } \frac{\pi}{2} = \cos(\frac{\pi}{2}) + i \sin(\frac{\pi}{2}) = i$

$w_3 = 1 \text{ cis } \frac{5\pi}{6} = \cos(\frac{5\pi}{6}) + i \sin(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_4 = 1 \text{ cis } \frac{7\pi}{6} = \cos(\frac{7\pi}{6}) + i \sin(\frac{7\pi}{6}) = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$w_5 = 1 \text{ cis } \frac{3\pi}{2} = \cos(\frac{3\pi}{2}) + i \sin(\frac{3\pi}{2}) = -i$

$w_6 = 1 \text{ cis } \frac{11\pi}{6} = \cos(\frac{11\pi}{6}) + i \sin(\frac{11\pi}{6}) = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

$\Rightarrow \boxed{\mathcal{S} = \{-i; i; \frac{\sqrt{3}}{2} + \frac{1}{2}i; -\frac{\sqrt{3}}{2} + \frac{1}{2}i; -\frac{\sqrt{3}}{2} - \frac{1}{2}i; \frac{\sqrt{3}}{2} - \frac{1}{2}i\}}$

b) dans $\mathbb{C}[x]$: $\boxed{(x+i)(x-i)\left(x - \frac{\sqrt{3}}{2} - \frac{1}{2}i\right)\left(x - \frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(x + \frac{\sqrt{3}}{2} + \frac{1}{2}i\right)\left(x + \frac{\sqrt{3}}{2} - \frac{1}{2}i\right)}$

dans $\mathbb{R}[x]$: $\boxed{(x^2+1)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)}$

Exercice 5.

$z_1 = 4 + 3i \quad \Rightarrow \quad z_2 = 4 - 3i$ est une solution

$(z - 4 - 3i)(z - 4 + 3i) = (z^2 - 8z + 25)$

$$\begin{array}{r} z^3 - 12z^2 + 57z - 100 = \left(z^2 - 8z + 25\right)(z - 4) \\ \underline{-z^3 \quad + 8z^2 - 25z} \\ \quad -4z^2 + 32z - 100 \\ \quad \underline{4z^2 - 32z + 100} \\ \quad 0 \end{array}$$

$\Rightarrow \boxed{\mathcal{S} = \{4; 4 + 3i, 4 - 3i\}}$

Exercice 6.

a) $\frac{\pi}{30} + \frac{2\pi}{5} = \frac{13\pi}{30} \qquad \frac{13\pi}{30} + \frac{2\pi}{5} = \frac{25\pi}{30} = \frac{5\pi}{6} \qquad \frac{5\pi}{6} + \frac{2\pi}{5} = \frac{37\pi}{30}$

$\frac{37\pi}{30} + \frac{2\pi}{5} = \frac{49\pi}{30}$

$$\Rightarrow \boxed{w_1 = 1\text{cis}\left(\frac{13\pi}{30}\right),\ w_2 = 1\text{cis}\left(\frac{5\pi}{6}\right),\ w_3 = 1\text{cis}\left(\frac{37\pi}{30}\right),\ w_4 = 1\text{cis}\left(\frac{49\pi}{30}\right)}$$

b) $a = 1^5 \cdot \cos\left(\frac{\pi}{6}\right) = \boxed{\frac{\sqrt{3}}{2}}$ et $b = 1^5 \cdot \sin\left(\frac{\pi}{6}\right) = \boxed{\frac{1}{2}}$